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The Improving Mathematics Education in Schools (TIMES) Project

ALGEBRAIC EXPRESSIONS

A guide for teachers - Year 7

NUMBER AND ALGEBRA : Module 23

June 2011

YEAR 7

Algebraic Expressions

(Number and Algebra : Module 23)

For teachers of Primary and Secondary Mathematics

510

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NUMBER AND ALGEBRA
Module 23

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YEAR 7

ALGEBRAIC EXPRESSIONS

ASSUMED KNOWLEDGE

- Fluency with addition, subtraction, multiplication and division of whole numbers and fractions.
- Ability to apply the any-order principle for multiplication and addition (commutative law and associative law) for whole numbers and fractions.
- Familiarity with the order of operation conventions for whole numbers.

MOTIVATION

Algebra is a fascinating and essential part of mathematics. It provides the written language in which mathematical ideas are described.

Many parts of mathematics are initiated by finding patterns and relating to different quantities. Before the introduction and development of algebra, these patterns and relationships had to be expressed in words. As these patterns and relationships became more complicated, their verbal descriptions became harder and harder to understand. Our modern algebraic notation greatly simplifies this task.

A well-known formula, due to Einstein, states that $E = mc^2$. This remarkable formula gives the relationship between energy, represented by the letter E, and mass, represented by letter m. The letter c represents the speed of light, a constant, which is about 300 000 000 metres per second. The simple algebraic statement $E = mc^2$ states that some matter is converted into energy (such as happens in a nuclear reaction), then the amount of energy produced is equal to the mass of the matter multiplied by the square of the speed of light. *You* can see how compact the formula is compared with the verbal description.

We know from arithmetic that $3 \times 6 + 2 \times 6 = 5 \times 6$. We could replace the number 6 in this statement by any other number we like and so we could write down infinitely many such statements. All of these can be captured by the algebraic statement $3x + 2x = 5x$, for any number x. Thus algebra enables us to write down general statements clearly and concisely.

The development of mathematics was significantly restricted before the 17th century by the lack of efficient algebraic language and symbolism. How this notation evolved will be discussed in the History section.

CONTENT

USING PRONUMERALS

In algebra we are doing arithmetic with just one new feature – we use letters to represent numbers. Because the letters are simply stand-ins for numbers, arithmetic is carried out exactly as it is with numbers. In particular the laws of arithmetic (commutative, associative and distributive) hold.

For example, the identities

$2 + x = x + 2$ $\qquad$ $2 \times x = x \times 2$

$(2 + x) + y = 2 + (x + y)$ $\qquad$ $(2 \times x) \times y = 2 \times (x \times y)$

$6(3x + 1) = 18x + 6$

hold when x and y are any numbers at all.

In this module we will use the word **pronumeral** for the letters used in algebra. We choose to use this word in school mathematics because of confusion that can arise from the words such as 'variable'. For example, in the formula $E = mc^2$, the pronumerals E and m are variables whereas c is a constant.

Pronumerals are used in many different ways. For example:

- **Substitution**: 'Find the value of $2x + 3$ if $x = 4$.' In this case the pronumeral is given the value 4.
- **Solving an equation**: 'Find x if $2x + 3 = 8$.' Here we are seeking the value of the pronumeral that makes the sentence true.
- **Identity**: 'The statement of the commutative law: $a + b = b + a$.' Here a and b can be any real numbers.
- **Formula**: 'The area of a rectangle is $A = lw$.' Here the values of the pronumerals are connected by the formula.
- **Equation of a line or curve**: 'The general equation of the straight line is $y = mx + c$.' Here m and c are parameters. That is, for a particular straight line, m and c are fixed.

In some languages other than English, one distinguishes between 'variables' in functions and 'unknown quantities' in equations ('incógnita' in Portuguese/Spanish, 'inconnue' in French) but this does not completely clarify the situation. The terms such as variable and parameter cannot be precisely defined at this stage and are best left to be introduced later in the development of algebra.

An **algebraic expression** is an expression involving numbers, parentheses, operation signs and pronumerals that becomes a number when numbers are substituted for the pronumerals. For example $2x + 5$ is an expression but $+) \times$ is not.

Examples of algebraic expressions are:

$3x + 1$ and $5(x^2 + 3x)$

As discussed later in this module the multiplication sign is omitted between letters and between a number and a letter. Thus substituting $x = 2$ gives

$3x + 1 = 3 \times 2 + 1 = 7$ and $5(x^2 + 3x) = 5(2^2 + 3 \times 2) = 30.$

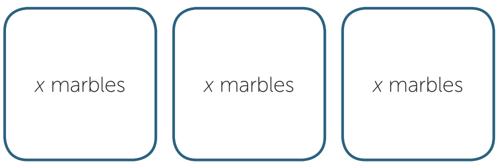
In this module, the emphasis is on expressions, and on the connection to the arithmetic that students have already met with whole numbers and fractions. The values of the pronumerals will therefore be restricted to the whole numbers and non-negative fractions.

Changing words to algebra

In algebra, pronumerals are used to stand for numbers. For example, if a box contains x stones and *you* put in five more stones, then there are $x + 5$ stones in the box. You may or may not know what the value of x is (although in this example we do know that x is a whole number).

- Joe has a pencil case that contains an unknown number of pencils. He has three other pencils. Let x be the number of pencils in the pencil case. Then Joe has $x + 3$ pencils altogether.

- Theresa has a box with least 5 pencils in it, and 5 are removed. We do not know how many pencils there are in the pencil case, so let z be the number of pencils in the box. Then there are $z - 5$ pencils left in the box.
- There are three boxes, each containing the same number of marbles. If there are x marbles in each box, then the total number of marbles is $3 \times x = 3x$.



- If n oranges are to be divided amongst 5 people, then each person receives $\frac{n}{5}$ oranges. (Here we assume that n is a whole number. If each person is to get a whole number of oranges, then n must be a multiple of 5.)

The following table gives us some meanings of some commonly occurring algebraic expressions.

$x + 3$	• The sum of x and 3 • 3 added to x, or x is added to 3 • 3 more than x, or x more than 3
$x - 3$	• The difference of x and 3 (where $x \geq 3$) • 3 is subtracted from x • 3 less than x • x minus 3
$3 \times x$	• The product of 3 and x • x is multiplied by 3, or 3 is multiplied by x
$x \div 3$	• x divided by 3 • the quotient when x is divided by 3
$2 \times x - 3$	• x is first multiplied by 2, then 3 is subtracted
$x \div 3 + 2$	• x is first divided by 3, then 2 is added

NOTATIONS AND LAWS

Expressions with zeroes and ones

Zeroes and ones can often be eliminated entirely. For example:

$x + 0 = x$ (Adding zero does not change the number.)

$1 \times x = x$ (Multiplying by one does not change the number.)

Algebraic notation

In algebra there are conventional ways of writing multiplication, division and indices.

Notation for multiplication

The $\times$ sign between two pronumerals or between a pronumeral and a number is usually omitted. For example, $3 \times x$ is written as $3x$ and $a \times b$ is written as ab. We have been following this convention earlier in this module.

It is conventional to write the number first. That is, the expression $3 \times a$ is written as $3a$ and not as $a3$.

Notation for division

The division sign ÷ is rarely used in algebra. Instead the alternative fraction notation for division is used. We recall 24 ÷ 6 can be written as $\frac{24}{6}$. Using this notation, x divided by 5 is written as $\frac{x}{5}$, not as $x \div 5$.

Index notation

$x \times x$ is written as x^2 and $y \times y \times y$ is written as y^3.

EXAMPLE

Write each of the following using concise algebraic notation.

a A number x is multiplied by itself and then doubled.

b A number x is squared and then multiplied by the square of a second number y.

c A number x is multiplied by a number y and the result is squared.

SOLUTION

a $x \times x \times 2 = x^2 \times 2 = 2x^2$.

b $x^2 \times y^2 = x^2y^2$.

c $(x \times y)^2 = (xy)^2$ which is equal to x^2y^2

Summary

- $2 \times x$ is written as $2x$
- $x \times y$ is written as xy or yx
- $x \times y \times z$ is written as xyz
- $x \times x$ is written as x^2
- $4 \times x \times x \times x = 4x^3$
- $x \div 3$ is written as $\frac{x}{3}$
- $x \div (yz)$ is written as $\frac{x}{yz}$
- $x^1 = x$ (the first power of x is x)
- $x^0 = 1$
- $1x = x$
- $0x = 0$

SUBSTITUTION

Assigning values to a pronumeral is called **substitution**.

EXAMPLE

If $x = 4$, what is the value of:

a $5x$ b $x + 3$ c $x - 1$ d $\frac{x}{2}$

SOLUTION

a $5x = 5 \times 4 = 20$ b $x + 3 = 4 + 3 = 7$ c $x - 1 = 4 - 1 = 3$ d $\frac{x}{2} = \frac{4}{2} = 2$

EXAMPLE

If $x = 6$, what is the value of:

a $3x + 4$ b $2x + 3$ c $2x - 5$

d $\frac{x}{2} - 2$ e $\frac{x}{3} + 2$

SOLUTION

a $3x + 4 = 3 \times 6 + 4 = 22$ b $2x + 3 = 2 \times 6 + 3 = 15$

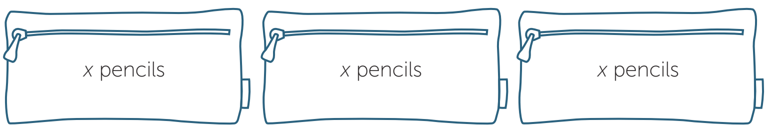
c $2x - 5 = 2 \times 6 - 5 = 7$ d $\frac{x}{2} - 2 = \frac{6}{2} - 2 = 1$

e $\frac{x}{3} + 2 = \frac{6}{3} + 2 = 4$

ADDING AND SUBTRACTING LIKE TERMS

Like terms

If you have 3 pencil case with the same number x of pencils in each, you have $3x$ pencils altogether.



If there are 2 more pencil cases with x pencils in each, then you have $3x + 2x = 5x$ pencils altogether. This can be done as the number of pencils in each case is the same. The terms $3x$ and $2x$ are said to be **like terms**.

Consider another example. If Jane has x packets of chocolates each containing y chocolates, then she has $x \times y = xy$ chocolates.

If David has twice as many chocolates as Jane, he has $2 \times xy = 2xy$ chocolates.

Together they have $2xy + xy = 3xy$ chocolates.

The terms $2xy$ and xy are **like terms**. Two terms are called **like terms** if they involve exactly the same pronumerals and each pronumeral has the same index.

EXAMPLE

Which of the following pairs consists of like terms:

a $3x$, $5x$ b $4x^2$, $8x$ c $4x^2y$, $12x^2y$

SOLUTION

a $5x$ and $3x$ are like terms.

b $4x^2$ and $8x$ are not like terms since the powers of x are different.

c $4x^2y$, $12x^2y$ are like terms

Adding and subtracting like terms

The distributive law explains the addition and subtraction of like terms. For example:

$$2xy + xy = 2 \times xy + 1 \times xy = (2 + 1)xy = 3xy$$

The terms $2x$ and $3y$ are not like terms because the pronumerals are different. The terms $3x$ and $3x^2$ are not like terms because the indices are different. For the sum $6x + 2y + 3x$, the terms $6x$ and $3x$ are like terms and can be added. There are no like terms for $2y$, so by using the commutative law for addition the sum is

$$6x + 2y + 3x = 6x + 3x + 2y = 9x + 2y.$$

The **any-order principle for addition** is used for the adding like terms.

Because of the commutative law and the associative law for multiplication (any-order principle for multiplication) the order of the factors in each term does not matter.

$$5x \times 3y = 15xy = 15yx$$

$$6ab \times 3b^2a = 18a^2b^3 = 18b^3a^2$$

Like terms can be added and subtracted as shown in the example below.

EXAMPLE

Simplify each of the following by adding or subtracting like terms:

a $2x + 3x + 5x$ b $3xy + 2xy$ c $4x^2 - 3x^2$

d $2x^2 + 3x + 4x$ e $4x^2y - 3x^2y + 3xy^2$

SOLUTION

a $2x + 3x + 5x = 10x$

b $3xy + 2xy = 5xy$

c $4x^2 - 3x^2 = x^2$

d $2x^2 + 3x + 4x = 2x^2 + 7x$

e $4x^2y - 3x^2y + 3xy^2 = x^2y + 3xy^2$

THE USE OF BRACKETS

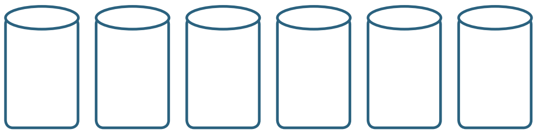
Brackets fulfill the same role in algebra as they do in arithmetic. Brackets are used in algebra in the following way.

> 'Six is added to a number and the result is multiplied by 3.'

Let x be the number. Then the expression is $(x + 6) \times 3$. We now follow the convention that the number is written at the front of the expression $(x + 6)$ without a multiplication sign. The preferred form is thus $3(x + 6)$.

EXAMPLE

For a party, the host prepared 6 tins of chocolate balls, each containing n chocolate balls.



a The host places two more chocolates in each tin. How many chocolates are there altogether in the tins now?

b If $n = 12$, that is, if there were originally 12 chocolates in each box, how many chocolates are there altogether in the tins now?

SOLUTION

a The number of chocolates in each tin is now $n + 2$. There are 6 tins, and therefore there are $6(n + 2)$ chocolates in total.

b If $n = 12$, the total number of chocolates is $6(n + 2) = 6(12 + 2) = 6 \times 14 = 84$.

EXERCISE 1

Each crate of bananas contains n bananas. Three bananas are removed from each crate.

a How many bananas are now in each crate?

b If there are 7 crates, how many bananas are there now in total?

EXERCISE 2

Five extra seats are added to each row of seats in a theatre. There were s seats in each row and there are 20 rows of seats. How many seats are there now in total?

Use of brackets and powers

The following example shows the importance of following the conventions of order of operations when working with powers and brackets.

- $2x^2$ means $2 \times x^2$
- $(2x)^2 = 2x \times 2x = 4x^2$

EXAMPLE

For $x = 3$, evaluate each of the following:

a $2x^2$ **b** $(2x)^2$

SOLUTION

a $2x^2 = 2 \times x \times x = 2 \times 3 \times 3 = 18$

b $(2x)^2 = (2 \times x)^2 = (2 \times 3)^2 = 36$

EXERCISE 3

Evaluate each expression for $x = 3$.

a $10x^3$ **b** $(10x)^3$

MULTIPLYING TERMS

Multiplying algebraic terms involves the **any-order property** of multiplication discussed for whole numbers.

The following shows how the any-order property can be used

$$\begin{aligned} 3x \times 2y \times 2xy &= 3 \times x \times 2 \times y \times 2 \times x \times y \\ &= 3 \times 2 \times 2 \times x \times x \times y \times y \\ &= 12x^2y^2 \end{aligned}$$

EXAMPLE

Simplify each of the following:

a $5 \times 2a$ **b** $3a \times 2a$ **c** $5xy \times 2xy$

SOLUTION

a $5 \times 2a = 10a$

b $3a \times 2a = 3 \times a \times 2 \times a = 6a^2$

c $5xy \times 2xy = 5 \times x \times y \times 2 \times x \times y = 10x^2y^2$

With practice no middle steps are necessary.

For example, $2a \times 3a \times 2a^2 = 12a^4$

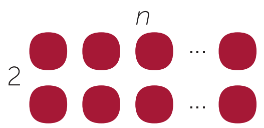
ARRAYS AND AREAS

ARRAYS

Arrays of dots have been used to represent products in the module *Multiplication of Whole Numbers.*

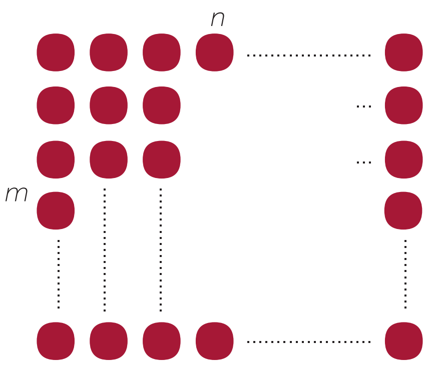
For example $2 \times 6 = 12$ can be represented by 2 rows of 6 dots.

The diagram below represents two rows with some number of dots.



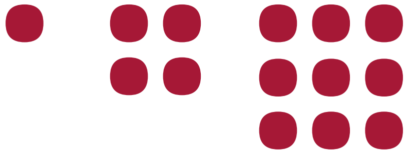
Let n be the number of dots in each row. Then there are $2 \times n = 2n$ dots.

If an array is m dots by n dots then there are mn dots. If the array is $m \times n$ then by convention we have m rows and n columns. We can represent the product $m \times n = mn$ by such an array.



EXAMPLE

The diagram shows squares formed by dots.
The pattern goes on forever.
How many dots are there in the n^{th} diagram?



SOLUTION

In the 1st diagram there are $1 \times 1 = 1^2$ dots.

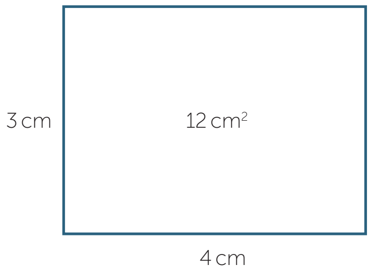
In the 2nd diagram there are $2 \times 2 = 2^2$ dots.

In the 3rd diagram there are $3 \times 3 = 3^2$ dots.

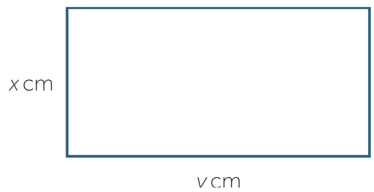
In the nth diagram there will be $n \times n = n^2$ dots.

Areas

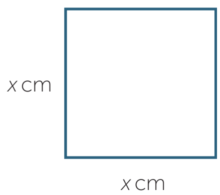
We can also represent a product like 3×4 by a rectangle.
The area of a 3 cm by 4 cm rectangle is 12 cm^2.



The area of a x cm by y cm rectangle is $x \times y = xy$ cm^2.
(x and y can be any positive numbers.)

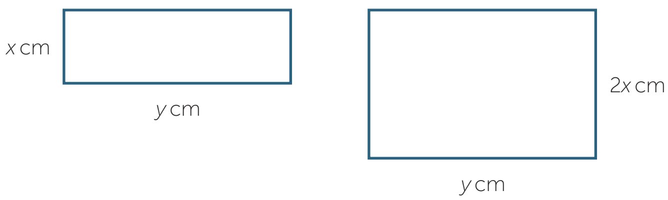


The area of a x cm by x cm square is x^2 cm^2.
(x can be any positive number.)



EXAMPLE

Find the total area of the two rectangles in terms of x and y.



SOLUTION

The area of the rectangle to the left is xy cm^2 and the area of the rectangle to the right is $2xy$ cm^2. Hence the total area is $xy + 2xy = 3xy$ cm^2.

NUMBER PATTERNS

Some simple statements with numbers demonstrate the convenience of algebra.

EXAMPLE

The n^{th} positive even number is $2n$.

a What is the square of the n^{th} positive even number?

b If the n^{th} positive even number is doubled what is the result?

SOLUTION

a The square of the n^{th} positive even number is $(2n)^2 = 4n^2$

b The double of the n^{th} positive even number is $2 \times 2n = 4n$.

EXERCISE 4

a If b is even, what is the next even number?

b If a is a multiple of three, what are the next two multiples of 3?

c If n is odd and $n \geq 3$, what is the previous odd number?

EXERCISE 5

Think of a 'number'. Let this number be x.

Write the following using algebra to see what *you* get.

- Multiply the number *you* thought of by 2 and subtract 5.
- Multiply the result by 3.
- Add 15.
- Subtract 5 times the number *you* first thought of.

EXERCISE 6

Show that the sum of the first n odd numbers is n^2.

ALGEBRAIC FRACTIONS

Quotients of expressions involving pronumerals often occur. We call them **algebraic fractions** we will meet this again in the module, *Special Expansions and Algebraic Fractions*.

EXAMPLE

Write each of the following in algebraic notation.

a A number is divided by 5, and 6 is added to the result.

b Five is added to a number, and the result is divided by 3.

SOLUTION

a Let x be the number.

Dividing by 5 gives $\frac{x}{5}$.

Adding 6 to this result gives $\frac{x}{5} + 6$.

b Let the number be x.

Adding 5 gives $x + 5$.

Dividing this by 3 gives $\frac{x+5}{3}$.

Notice that the vinculum acts as a bracket.

EXAMPLE

If $x = 10$, find the value:

a $\frac{x}{2}$ b $\frac{x}{5} + 3$ c $\frac{x+4}{3}$ d $\frac{x-4}{4}$

SOLUTION

a $\frac{x}{2} = \frac{10}{2}$

$= 5$

b $\frac{x}{5} + 3 = \frac{10}{5} + 3$

$= 5$

c $\frac{x+4}{3} = \frac{10+4}{3}$

$= \frac{14}{3}$

$= 4\frac{2}{3}$

d $\frac{x-4}{4} = \frac{10-4}{4}$

$= \frac{6}{4}$

$= \frac{3}{2}$

EXAMPLE

A vat contains n litres of oil. Forty litres of oil are then added to the vat.

a How many litres of oil are there now in the vat?

b The oil is divided into 50 containers. How much oil is there in each container?

SOLUTION

a There is a total of $n + 40$ litres of oil in the vat.

b There are $\frac{n+40}{50}$ litres of oil in each container.

EXERCISE 7

A shed contains n tonnes of coal. An extra 1000 tonnes are then added.

a How many tonnes of coal are there in the shed now?

b It is decided to ship the coal in 10 equal loads. How many tonnes of coal are there in each load?

EXPANDING BRACKETS AND COLLECTING LIKE TERMS

Expanding brackets

Numbers obey the distributive laws for multiplication over addition and subtraction. For example:

$3 \times (4 + 5) = 3 \times 4 + 3 \times 5$ $\qquad$ $7 \times (6 - 3) = 7 \times 6 - 7 \times 3$

The distributive laws for division over addition and subtraction also hold as shown.
For example:

$(8 + 6) \div 2 = 8 \div 2 + 6 \div 2$ and $\frac{9-7}{3} = \frac{9}{3} - \frac{7}{3}$

As with adding like terms and multiplying terms, the laws that apply to arithmetic can be extended to algebra. This process of rewriting an expression to remove brackets is usually referred to as **expanding brackets**.

EXAMPLE

Use the distributive law to rewrite these expressions without brackets.

a $5(x - 4)$ $\qquad$ b $4(3x + 2)$ $\qquad$ c $6(4 - 2x)$

SOLUTION

a $5(x - 4) = 5x - 20$ $\qquad$ b $4(3x + 2) = 12x + 8$

c $6(4 - 2x) = 24 - 12x$

EXAMPLE

Use the distributive law to rewrite these expressions without brackets.

a $\frac{3x+1}{3}$

b $\frac{2-x}{2}$

SOLUTION

a $\frac{3x+1}{3} = x + \frac{1}{3}$

b $\frac{2-x}{2} = 1 - \frac{x}{2}$

Collecting like terms

After brackets have been expanded like terms can be collected.

EXAMPLE

Expand the brackets and collect like terms:

a $2(x - 6) + 5x$

b $3 + 3(x - 1)$

c $3(2x + 4) + 6(x - 1)$

SOLUTION

a $2(x - 6) + 5x = 2x - 12 + 5x = 7x - 12$

b $3 + 3(x - 1) = 3 + 3x - 3 = 3x$

c $3(2x + 4) + 6(x - 1) = 6x + 12 + 6x - 6 = 12x + 6$

EXERCISE 8

Expand the brackets and collect like terms:

a $5(x + 2) + 2(x - 3)$

b $2(7 + 5x) + 4(x + 6)$

c $3(2x + 7) + 2(x - 5)$

LINKS FORWARD

A sound understanding of algebra is essential for virtually all areas of mathematics.

The introduction to algebra is continued in the modules, *Negatives and the Index Laws in Algebra*, *Special Expansions and Algebraic Fractions* and *Fractions and the Index Laws in Algebra*.

HISTORY

It was only in the 17th century that algebraic notation similar to that used today was introduced. For example, the notation used by Descartes (*La Geometrie*, 1637) and Wallis (1693) was very close to modern notation. However, algebra has a very long history.

There are examples of the ancient Egyptians working with algebra. About 1650 BC, the Egyptian scribe Ahmes made a transcript of even more ancient mathematical scriptures dating to the reign of the Pharaoh Amenemhat III. In 1858 the Scottish antiquarian, Henry Rhind, came into possession of Ahmes's papyrus. The papyrus is a scroll 33 cm wide and about 5.25m long filled with mathematical problems. One of the problems is as follows:

> *100 measures of corn must be divided among 5 workers, so that the second worker gets as many measures more than the first worker, as the third gets more than the second, as the fourth gets more than the third, and as the fifth gets more than the fourth. The first two workers shall get seven times fewer measures of corn than the three others. How many measures of corn shall each worker get? (The answer involves fractional measures of corn. Answer:* $1\frac{2}{3}$ *,* $10\frac{5}{6}$ *, 20,* $29\frac{1}{6}$ *and* $38\frac{1}{3}$ *measures.)*

Euclid (circa 300 BC) dealt with algebra in a geometric way and algebraic problems are solved without using algebraic notation of any form. Diophantus (circa 275 AD) who is sometimes called the father of algebra, produced a work largely devoted to the ideas of the subject.

Chinese and Indian authors wrote extensively on algebraic ideas and achieved a great deal in the solution of equations. The earliest Chinese text with algebraic ideas is *The Nine Chapters on the Mathematical Art*, written around 250 BC. A later text is *Shu-shu chiu-chang*, or *Mathematical Treatise in Nine Sections*, which was written by the wealthy governor and minister Ch'in Chiu-shao (circa 1202 – circa 1261 AD)

In approximately 800 BC an Indian mathematician Baudhayana, in his *Baudhayana Sulba Sutra*, discovered Pythagorean triples algebraically, and found geometric solutions of linear equations and quadratic equations of the forms, $ax^2 = c$ and of $ax^2 + bx = c$.

In 680 AD the Indian mathematician Brahmagupta, in his treatise *Brahma Sputa Siddhanta*, worked with quadratic equations and determined rules for solving linear and quadratic equations. He discovered that quadratic equations can have two roots, including both negative and irrational roots.

Indian mathematician Aryabhata, in his treatise *Aryabhatiya*, obtains whole-number solutions to linear equations by a method equivalent to the modern one.

Arab and Persian mathematicians had an interest in algebra and their ideas flowed into Europe. One algebraist of special prominence was al-Khwarizmi, whose al-jabr w'al muqabalah (circa 825 AD) gave the discipline its name and gave the first systematic treatment of algebra.

The works of al-Khwarizmi were translated into European languages by several different translators during the twelfth century and so his and other Arab writers' work were well known in Europe.

Fibonacci was the greatest European writer on algebra during the middle ages and his work *Liber Quadratorum* (circa 1225 AD) includes many different ingenious ways of solving equations.

Cardano, Tartaglia (16th century), Vieta (16th – 17th century) and others developed the ideas and notation of algebra. The *Ars Magna* (Latin: "The Great Art") is an important book on Algebra written by Gerolamo Cardano. It was first published in 1545 under the title *Artis Magnæ, Sive de Regulis Algebraicis Liber Unus* (*Book number one about The Great Art*, or *The Rules of Algebra*). There was a second edition in Cardano's lifetime, published in 1570. It is considered one of the three greatest scientific treatises of the Renaissance. The book included the solutions to the cubic and quartic equations. The solution to one particular case of the cubic, $x^3 + ax = b$ (in modern notation), was communicated to him by Niccolò Fontana Tartaglia, and the quartic was solved by Cardano's student Lodovico Ferrari.

Development of algebraic notation

Here are some of the different notations used from the Middle Ages onwards together with their modern form. They reveal how useful modern algebraic notation is.

Trouame.1.n^0.che giōto al suo q̄drat^0 facia.12
$x + x^2 = 12$ – Pacioli (1494)

4 Se. –51 Pri. –30 N. dit is ghelijc $45\frac{3}{5}$
$4x^4 - 51x - 30 = 45\frac{3}{5}$ – Vander Hoecke (1514)

cub p: 6 reb æq̄lis 20.
$x^3 + 6x = 20$ – Cardano (1545)

1 Q C –15 Q Q + 85 C –225 Q + 274 N æquator 120.
$x^6 - 15x^4 + 85x^3 - 225x^2 + 274x = 120$ – Vieta (c. 1590)

aaa – 3.*bba* ======= + 2.*ccc*
$x^3 - 3b^2x = 2c^3$. – Harriot (1631)

REFERENCES

A History of Mathematics: An Introduction, 3rd Edition, Victor J. Katz, Addison-Wesley, (2008)

ANSWERS TO EXERCISES

EXERCISE 1

a $n - 3$ bananas in each crate **b** $7(n - 3)$ bananas in total

EXERCISE 2

a $20(s + 5)$ seats in total

EXERCISE 3

a 270 **b** 27 000

EXERCISE 4

a $b + 2$ **b** $a + 3, a + 6$ **c** $n - 2$

EXERCISE 5

$x \longrightarrow 2x - 5 \longrightarrow 3(2x - 5) = 6x - 15 \longrightarrow 6x \longrightarrow 6x - 5x = x$

EXERCISE 6

Sum of the first n odd numbers is: $1 + 3 + 5 + ... + (2n - 5) + (2n - 3) + (2n - 1)$

Reverse the sum: $(2n - 1) + (2n - 3) + (2n - 5) + ... + 5 + 3 + 1$

Add to the original sum pairing terms yields

$(2n - 1 + 1) + (2n - 3 + 3) + ... + (3 + 2n - 3) + (1 + 2n - 1) = 2n \times n = 2n^2$

The sum is n^2

EXERCISE 7

a $n + 1000$ tonnes **b** $\frac{n + 1000}{10}$ tonnes

EXERCISE 8

a $7x + 4$ **b** $14x + 38$ **c** $8x + 11$

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